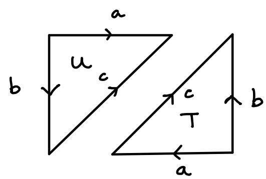
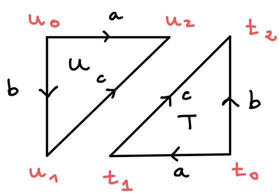
About orientations on a Δ-complex structure

Suppose we can write our space X as $Y/\sim$, where Y is a disjoint union of simplices and $\sim$ are some identifications of some of the faces. I said in the problem class that one can choose any orientations and this gives a Δ-complex structure on X. This is not true.

The correct statement is that one needs to give an orientation (i.e. an ordering of the vertices) on each simplex, so that these are compatible. For example, if we start with the following presentation of $\mathbb{RP}^2$:



we have to order the vertices of U and T so that these orderings agree after the identification $\sim$:

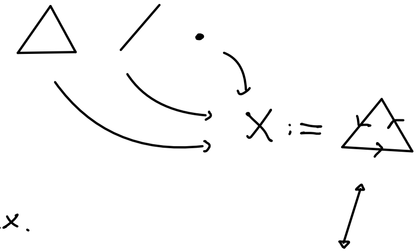


check: the arrow c goes from smaller to bigger ($u_2 > u_1$, $t_2 > t_1$) in both simplices

similarly for a, b

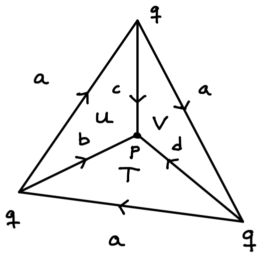
It is okay if an arrow goes from bigger to smaller, as long as it does so for any induced orientations, but then we would have to put a $-$ sign when taking the boundary map, which is confusing. Thus it is convenient to swap the direction of the arrow to go from smaller to bigger always.

In particular, there is no way to make this collection of maps

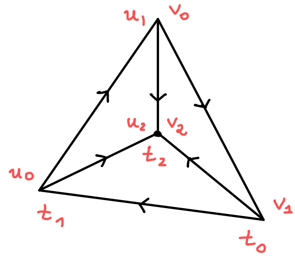


into a Δ-complex.

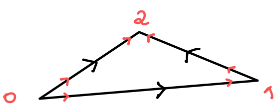
Note: this does not mean that X cannot be given such a structure. For example,

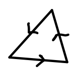


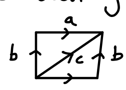
$\cong$ X with orderings



In general, note that, if one manages to swap the arrows for the identifications so that <u>any</u> 2-simplex looks like



(and not )

then there is a "canonical" orientation. That is why, when you see a sentence like "take the following Δ-complex structure : 

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It just means to take the canonical orientations, and not any orientations as I had said. It is true that taking any orientation will work out for the computation, but the correct thing is what I have explained here.

I wrote this in terms of arrows and 2-simplices because it is easier to visualize, but the analogue works in general, noting that "arrow" is just a choice of linear map from $[0,1]=\Delta_1$ to X

In particular, there is only <u>one</u> way to glue all k-dimensional faces of an n-simplex so that we get a Δ-complex structure

Pick an ordering of the vertices. For each k-face σ_i restricting the ordering to σ_i gives rise to a unique iso $f_i : \Delta_k \longrightarrow \sigma_i$, and one has to glue the σ_i according to these (so $x \in \sigma_i$ is glued to $y \in \sigma_j$ if $f_i^{-1}(x) = f_j^{-1}(y)$).

The associated chain complex is

$$\mathbb{Z}\sigma_n \xrightarrow{\partial_n} \mathbb{Z}\sigma_{n-1} \longrightarrow \cdots \longrightarrow \mathbb{Z}\sigma_1 \xrightarrow{\partial_1} \mathbb{Z}\sigma_0$$

where $\partial_k(\sigma_k) = \sum_{i=0}^{k} (-1)^i \cdot \sigma_{k-1} = \begin{cases} 0 \text{ if } k \text{ odd} \\ \sigma_{k-1} \text{ if } k \text{ even} \end{cases}$, so $H_k^{\Delta}(X) = \begin{cases} \mathbb{Z} \text{ if } k \text{ odd or } k=0 \\ 0 \text{ otherwise} \end{cases}$